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# Rationalisation of second law analysis of heat exchangers

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**Abstract**

The purpose of this paper is to review the various approaches to second law analysis and to present a rational method which satisfies the physical requirements. It is not the intention to review all previous work, but to present an approach that resolves some perceived inconsistencies and paradoxes. Some new relationships are derived, particularly for the local rate generation process, and for the nearly-balanced counterflow arrangement with a 'long' duty. It is also shown that the basic entropy generation relationship for gas flows is controlled by the flow Mach number, which is consistent with an extension of Shapiro's classical one-dimensional flow analysis of a compressible gas with friction and heat addition. 

## 1. Introduction

Much work on the second law analysis of heat exchangers has been conducted by Bejan [1], Sekulic [2], McClintock [3], Witte and Shamsundar [4], London and Shah [5], and others. One of the main differences between workers lies in how the entropy generation rate is non-dimensionalised, and examination of this is given in Section 3. The present analysis will be restricted to perfect gas flow on the basis that frictional entropy generation for liquids is very small in most situations.

## 2. Basics of entropy generation

We start with the first and second law statements for a one-dimensional heat transfer duct as given by Bejan [6], referring to Fig. 1:

$$\text{1st law:} \quad \dot{m}\,\mathrm{d}h = q'\,\mathrm{d}x \tag{1}$$

$$\text{2nd law:} \quad \mathrm{d}\dot{S}_{\mathrm{gen}} = \dot{m}\,\mathrm{d}s - \frac{q'\,\mathrm{d}x}{T+\Delta T} \geq 0 \tag{2}$$

The canonical thermodynamic relationship for entropy is

$$\mathrm{d}h = T\,\mathrm{d}s + \frac{\mathrm{d}p}{\rho}, \tag{3}$$

giving

$$\frac{\mathrm{d}h}{\mathrm{d}x} = T\frac{\mathrm{d}s}{\mathrm{d}x} + \frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} \tag{4}$$

Linking Eqs. (1)–(3) gives

$$\dot{S}'_{\mathrm{gen}} = \frac{\mathrm{d}\dot{S}_{\mathrm{gen}}}{\mathrm{d}x} = \frac{q'\Delta T}{T^2(1+\tau)} + \frac{\dot{m}}{\rho T}\left(-\frac{\mathrm{d}p}{\mathrm{d}x}\right) \tag{5}$$

where $\tau = \Delta T/T$, the dimensionless temperature difference.

A starting point for our understanding of the entropy generation process can be obtained by considering just the thermal component, and noting that

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**Nomenclature**

| | |
|---|---|
| $a$ | speed of sound |
| $A$ | parameter (dimensional) |
| $A_1$ | parameter |
| $A_2$ | parameter |
| $A_s$ | surface area |
| $A_c$ | flow area |
| $B$ | dimensionless heat flow (Eq. (24)) |
| $Be$ | Bejan number |
| $B_0$ | dimensionless parameter (Bejan, Eq. (73)) |
| $B_1$ | dimensionless parameter (Eq. (64)) |
| $B_2$ | dimensionless parameter (Eq. (68)) |
| $B_3$ | dimensionless parameter (Eq. (82)) |
| $c_p$ | specific heat at constant pressure |
| $C$ | heat capacity rate |
| $C_1$ | constant (Eq. (83)) |
| $C^*$ | ratio of heat capacity |
| $d_h$ | hydraulic diameter |
| $f$ | Fanning friction factor |
| $g$ | dimensionless mass velocity (Bejan) |
| $G$ | mass velocity |
| $j$ | Colburn factor |
| $L$ | flow length |
| $\dot{m}$ | mass flow number |
| $M$ | Mach number |
| $Ntu$ | number of thermal units |
| $N_s$ | entropy generation number based on heat capacity rate |
| $N_{s1}$ | entropy generation number based on heat flow |
| $Nu$ | Nusselt number |
| $p$ | pressure |
| $Pr$ | Prandlt number |
| $p_s$ | perimeter |
| $\dot{Q}$ | heat flow |
| $q'$ | heat gradient |
| $r_0$ | inner radius of tube |
| $R$ | ideal gas constant |
| $Re$ | Reynolds number |
| $St$ | Stanton number |
| $S'$ | entropy gradient |
| $\dot{S}_{gen}$ | entropy generation rate |
| $t$ | dimensionless temperature ratio, $T_{in}/T_{out}$ |
| $T$ | absolute temperature |
| $v$ | velocity |
| $x$ | axial distance |

*Greek symbols*

| | |
|---|---|
| $\alpha$ | heat transfer coefficient |
| $\varepsilon$ | effectiveness |
| $\rho$ | density |
| $\tau$ | dimensionless temperature difference $\Delta T/T$ |
| $\eta$ | dynamic viscosity |
| $\lambda$ | thermal conductivity |

*Subscripts*

| | |
|---|---|
| 1 | cold stream |
| 2 | hot stream |
| in | inlet |
| out | outlet |
| c | cold stream |
| h | hot stream |
| min | minimum |
| max | maximum |
| opt | optimum |
| ref | reference |

$$q' = \alpha p_s \Delta T \tag{6}$$

where $p_s$ is the surface perimeter. Then it is easily shown by substitution in Eq. (5) that for an incremental surface area $\Delta A_s$, the incremental entropy generation $\Delta\dot{S}_{gen}$ is given by

$$\Delta\dot{S}_{gen} = \frac{\alpha \Delta A_s \tau^2}{1+\tau} \tag{7}$$

Thus the entropy generation rate for the thermal component is proportional to the square of the dimensionless temperature difference $\tau$. The importance of this for cryogenic applications (low absolute temperature $T$) is clear.

Considering now two streams in a heat exchanger (see Fig. 1 for example), with hot and cold inlet temperatures $T_2$ and $T_1$, respectively, we can write

$$\mathrm{d}\dot{S}_{gen} = \dot{m}_1\,\mathrm{d}s_1 - \left(\frac{q'\,\mathrm{d}x}{T+\Delta T}\right)_1 + \dot{m}_2\,\mathrm{d}s_2 + \left(\frac{q'\,\mathrm{d}x}{T+\Delta T}\right)_2 - \dot{m}_1 R_1 \frac{\mathrm{d}p_1}{p_1} - \dot{m}_2 R_2 \frac{\mathrm{d}p_2}{p_2}, \tag{8}$$

which becomes, on integration,

$$\dot{S}_{gen} = (\dot{m}c_p)_1 \ln\left(\frac{T_{1out}}{T_1}\right) + (\dot{m}c_p)_2 \ln\left(\frac{T_{2out}}{T_2}\right) + (\dot{m}R)_1 \ln\left(\frac{p_1}{p_{1out}}\right) + (\dot{m}R)_2 \ln\left(\frac{p_2}{p_{2out}}\right), \tag{9}$$

Initial observation of this equation indicates that if the terminal temperatures $T_1$, $T_2$, $T_{1out}$, and $T_{2out}$ are fixed by process considerations such as a pinch condition, which implies fixed driving temperature differences, the first two terms are fixed, but the pressure drop contributions can be controlled by increasing the flow area in accordance with the core velocity equation [7]. This point will be further investigated later in the paper.

Some fundamental relationships linking entropy generation with heat exchange parameters are first re-examined for the case of zero pressure drop. Pressure drop is then taken into account in Section 4, allowing for optimisation, or entropy minimisation analysis.

## 3. Zero pressure drop

### 3.1. Balanced counterflow

We start with the case of balanced counterflow, referring to Fig. 1, for which the performance is described by the ε–Ntu relationship

$$\varepsilon = \frac{T_{1\mathrm{out}} - T_1}{T_2 - T_1} = \frac{T_2 - T_{2\mathrm{out}}}{T_2 - T_1} = \frac{Ntu}{1 + Ntu}, \tag{10}$$

which gives [8],

$$\dot{S}_{\mathrm{gen}} = \dot{m}c_{\mathrm{p}} \ln\left[\frac{\left(1 + \frac{T_1}{T_2}Ntu\right)\left(1 + \frac{T_2}{T_1}Ntu\right)}{(1 + Ntu)^2}\right]. \tag{11}$$

The most obvious way of non-dimensionalising this equation is to divide by the heat capacity rate $\dot{m}c_{\mathrm{p}}$ of each stream, giving the entropy generation number $N_{\mathrm{s}}$ [8]:

$$N_{\mathrm{s}} = \frac{\dot{S}_{\mathrm{gen}}}{\dot{m}c_{\mathrm{p}}} = \ln\left[\frac{\left(1 + \frac{T_1}{T_2}Ntu\right)\left(1 + \frac{T_2}{T_1}Ntu\right)}{(1 + Ntu)^2}\right], \tag{12}$$

or, in terms of effectiveness,

$$N_{\mathrm{s}} = \ln\left(\left(1 + \varepsilon\left(\frac{T_2}{T_1} - 1\right)\right)\left(1 - \varepsilon\left(1 - \frac{T_1}{T_2}\right)\right)\right). \tag{13}$$

This function is illustrated in Fig. 2, with ε as abscissa. As noted by Bejan, $N_{\mathrm{s}}$ approaches zero in two limits: $Ntu \rightarrow \infty$ (or $\varepsilon \rightarrow 1$), representing the ideal limit of zero driving temperature difference; and $Ntu$ (with $\varepsilon \rightarrow 0$). Note the symmetry of the function, reflected in the figure with a maximum entropy generation at $Ntu$ = 0.5. Bejan calls this behaviour the 'entropy generation paradox', and the $Ntu \rightarrow 0$ limit the 'vanishing heat exchanger limit'.

Similar forms of curve with maxima have been obtained by Sekulic [2] for other heat exchanger configurations.

The physical basis of this approach warrants some further attention. Firstly, looking at the 'low' limit, expressing $Ntu$ as

$$Ntu = \frac{St4L}{d_{\mathrm{h}}} \tag{14}$$

as $Ntu$ tends to zero, either $St$ (or heat transfer coefficient) tends to zero, which is not a realistic scenario, or the flow length $L$ tends to zero — for sensible values of hydraulic diameter. Thus this limit practically represents a vanishing flow length, with correspondingly a vanishing heat flow $\dot{Q}$, as noted by Bejan [9]. Further, the fixed (finite) capacity rate implies a finite flow area, if only to comply with the assumption of negligible pressure drop. This in turn requires that the lateral dimensions are finite, so we can say that the

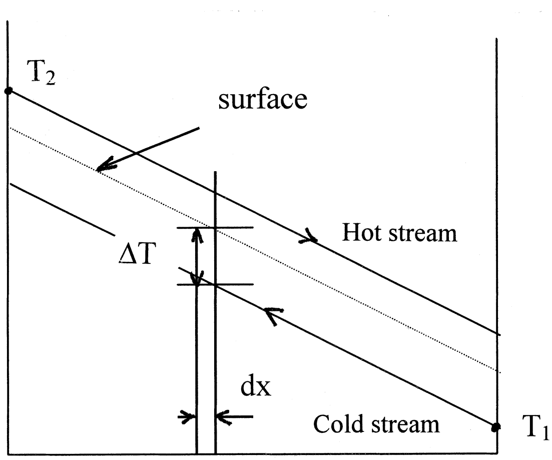


Fig. 1. Elemental surface in heat exchanger.

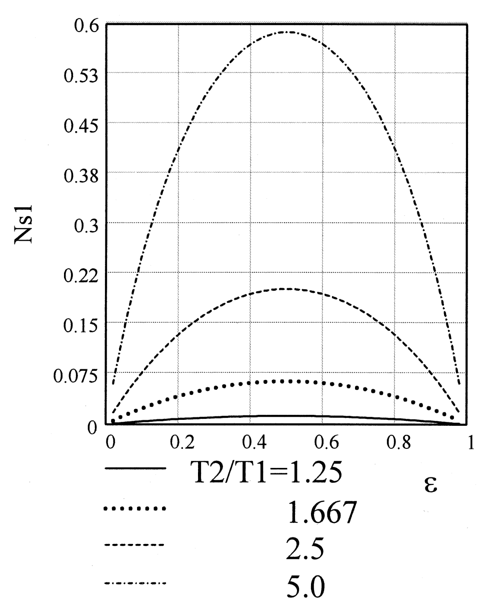


Fig. 2. Bejan's parameter $N_{\mathrm{s}}$ versus ε, for (1) $T_2/T_1 = 1.25$, (2) $T_2/T_1 = 1.667$, (3) $T_2/T_1 = 2.5$, (4) $T_2/T_1 = 5$.

exchanger only 'vanishes' in one dimension, which is not physically sensible. A truly vanishing heat exchanger has all dimensions vanishing, and hence zero flow rate by implication. Thus the fact of zero entropy generation has no practical significance for heat exchanger purposes. The upper, L tends to infinity, limit is the infinite area, or ideal limit and $\dot{Q} = \dot{Q}_{\max}$ in the usual notation.

These observations point to using the heat flow as a more appropriate means of non-dimensionalising the entropy generation, since the heat flow characterises the raison d'être of the exchanger. The most rational way in terms of exergy analysis would be to non-dimensionalise by $\dot{Q}/T_0$. This approach is taken by Witte and Shamsundar [4], and London and Shah [5]. A disadvantage is that it introduces a further temperature (the ambient $T_0$) into the analysis, in addition to the terminal temperatures $T_1$ and $T_2$, and complicates presentation. At this point it is worth noting that the basic Eq. (9) easily reduces to the form expressed independently by Witte and Shamsundar and London and Shah, by putting

$$(\dot{m}c_p)_1 = (\dot{m}c_p)_c = \frac{\dot{Q}}{T_{c\text{out}} - T_{c\text{in}}}, \tag{15a}$$

and

$$(\dot{m}c_p)_2 = (\dot{m}c_p)_h = \frac{\dot{Q}}{T_{h\text{in}} - T_{h\text{out}}}, \tag{15b}$$

so

$$\dot{S}_{\text{gen}} = \frac{\dot{Q}}{T_{c\text{out}} - T_{c\text{in}}} \ln\left(\frac{T_{c\text{out}}}{T_{c\text{in}}}\right) + \frac{\dot{Q}}{T_{h\text{in}} - T_{h\text{out}}} \ln\left(\frac{T_{h\text{out}}}{T_{h\text{in}}}\right), \tag{16}$$

and

$$\frac{\dot{S}_{\text{gen}}}{\dot{Q}} = \frac{1}{\bar{T}_c} - \frac{1}{\bar{T}_h}, \tag{17}$$

where $\bar{T}_c$ and $\bar{T}_h$ are the log-mean temperatures. This is the form found by Witte and Shamsundar, and London and Shah.

A further point of interest here is that Witte and Shamsundar derived Eq. (17) by interposing a reversible heat engine and heat pump between the two streams, and showed that the necessary heat interaction $\dot{Q}_0$ with the environment was equal to the lost net work $\dot{W}_{\text{rev}}$, or in other words the lost capability of the system to do work, represented by the Gouy–Stodola relationship:

$$\dot{W}_{\text{rev}} = T_0 \dot{S}_{\text{gen}}. \tag{18}$$

The maximum $N_s = \dot{S}_{\text{gen}}/\dot{m}c_p$ indicated in Fig. 2 thus corresponds to the maximum work rate, or power, that could be obtained from the system with given flow rates if the heat exchanger were replaced by a reversible power system, with perfect internal heat exchangers, leaving the terminal temperatures intact. This contrasts with the normal power generation scenario which utilises constant temperature flow streams and the *difference* in heat flow between them.

In the current approach we non-dimensionalise by $\dot{Q}/T_1$, in order to avoid the introduction of another variable ($T_0$), and call the revised entropy generation number $N_{s1}$, to avoid confusion with Bejan's $N_s$ based on $\dot{m}c_p$. Since the heat flow is

$$\dot{Q} = \dot{m}c_p \varepsilon (T_2 - T_1), \tag{19}$$

the entropy generation number becomes

$$N_{s1} = \frac{T_1 \dot{S}_{\text{gen}}}{\dot{Q}} = \frac{1}{\varepsilon(T_2/T_1 - 1)} \ln\left(\frac{(1 - T_2 Ntu/T_1)(1 + T_2 Ntu/T_1)}{(1 + Ntu)^2}\right). \tag{20}$$

$$N_{s1} = \frac{1}{\varepsilon(T_2/T_1 - 1)} \ln\left(\left(1 + \varepsilon\left(\frac{T_2}{T_1} - 1\right)\right) \times \left(1 - \varepsilon\left(1 - \frac{T_1}{T_2}\right)\right)\right) = \frac{N_s}{\varepsilon(T_2/T_1 - 1)}. \tag{21}$$

This function is shown in Fig. 3(a), for the same range of $T_2/T_1$ as Fig. 2. Note that Bejan's $N_s$ (the logarithmic term in the equation) has been divided by $\varepsilon(T_2/T_1 - 1)$ which is $\dot{Q}/\dot{m}c_p T_1$, to obtain $N_{s1}$. This dimensionless heat flow is proportional to effectiveness as shown in Fig. 4(a) for the same values of $T_2/T_1$. Its effect on $N_{s1}$ is more readily grasped by its reciprocal, Fig. 4(b), which is multiplied by Bejan's $N_s$ to obtain Fig. 3(a).

Summarising the above arguments, we see that the maximum entropy generation rate ($N_s$) for a fixed heat capacity rate $\dot{m}c_p$ occurs when the corresponding heat flow is one half of the maximum (i.e., at $\varepsilon = 0.5$). Interestingly, this corresponds to the case examined by Sekulic [2] relating the entropy generation to that of the adiabatic mixing of the two streams. Bejan's approach of non-dimensionalising by heat capacity rate would thus be appropriate to apply to a power generation or process situation in which the heat exchanger is a function in a given pair of flow streams but its heat load is not specified and the absolute entropy generation is to be controlled. The purpose of the exchanger is not then to exchange a given rate of

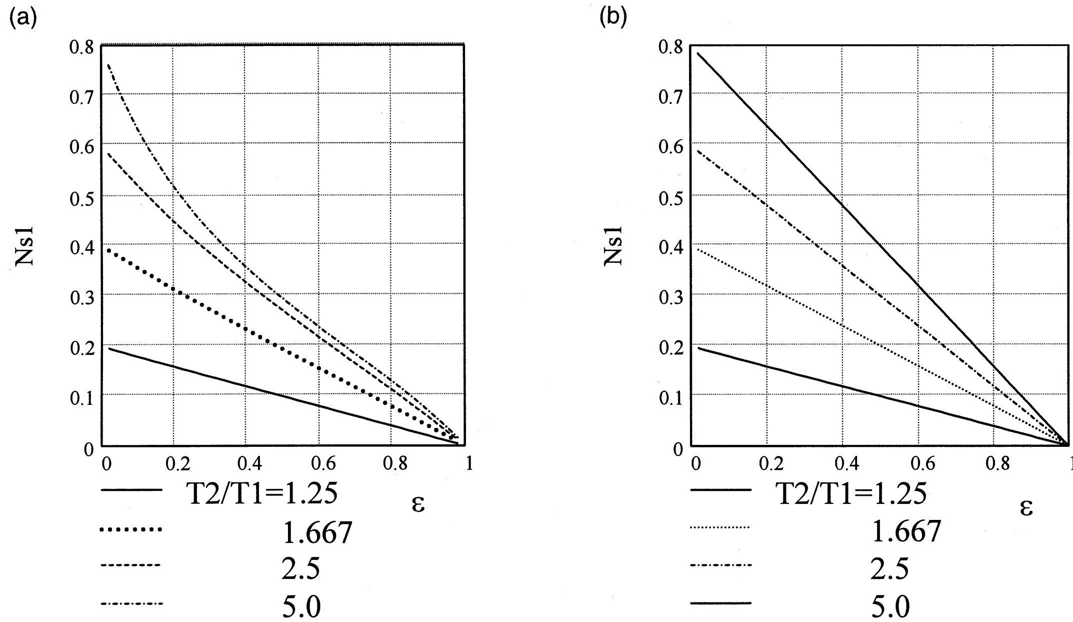


Fig. 3. (a) Entropy generation number $N_{s1}$ (Eq. (21)). (b) Entropy generation number $N_{s1}$. Simplification (Eq. (23)).

heat, but to enable the system to operate with given irreversibility. For arbitrary selection of ε, the heat flow is then determined by Eq. (19), and tends to zero in the 'vanishing heat exchanger limit' of $\varepsilon \to 0$, and to its maximum in the 'perfect' heat exchanger limit of $\varepsilon \to 1$.

Conversely, for process situations in which the heat load $\dot{Q}$ is given, the entropy generation rate ($N_{s1}$)

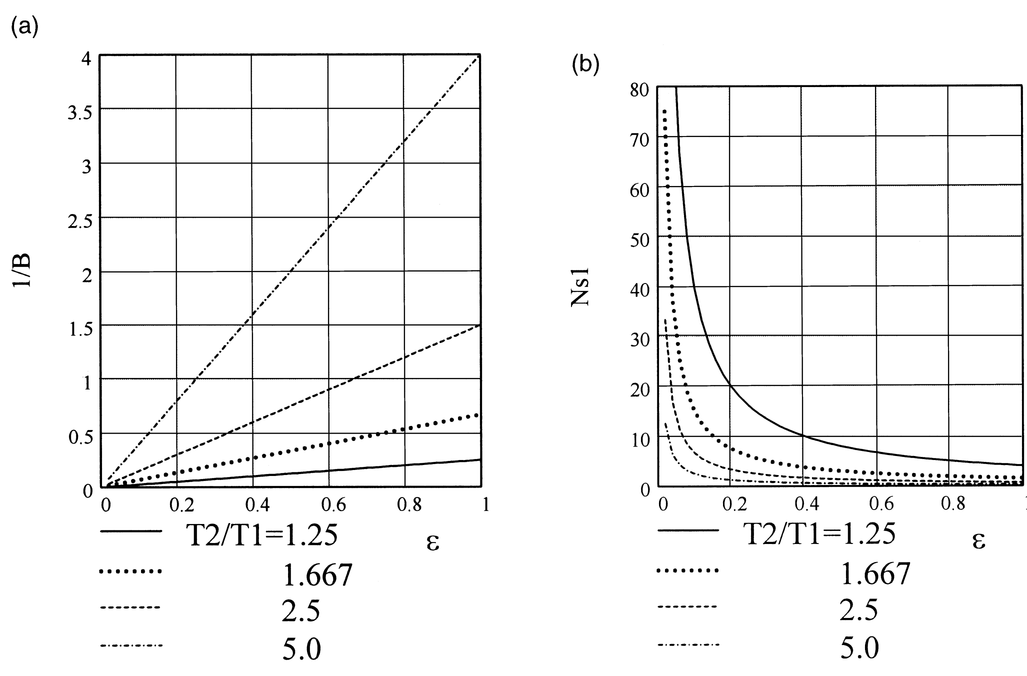


Fig. 4. (a) Parameter $\dot{Q}/\dot{m}c_pT_1$ $(= B)$ $(= \varepsilon/(T_2/T_1 - 1))$. (b) Parameter $\dot{M}c_p/\dot{Q}$ $(= 1/B)$ $(= (T_2/T_1 - 1)/\varepsilon)$.

decreases monotonically as $\varepsilon$ or $Ntu$ increases. Note that the formulation of $N_{s1}$ is such that for a given heat capacity rate $\dot{m}c_p$ and initial temperatures $T_1$ and $T_2$, the specification of heat load $\dot{Q}\,(<\dot{Q}_{max})$ directly determines $\varepsilon$ and hence $\dot{S}_{gen}$, as is evident from Eq. (21) since all four terminal temperatures are then fixed. If $\dot{Q}$ and temperature limits only are specified, the process designer has one degree of freedom ($\dot{m}c_p$ or $\varepsilon$), linked by Eq. (19).

The ratio $N_{s1}$ now behaves in a more intuitively reasonable way, completing the resolution of the 'paradox'. Bejan [10] states: "...we expect any heat transfer irreversibility to increase monotonically as heat exchanger area (or Ntu) decreases". These observations also apply to the unbalanced cases dealt with in the following section.

A simplification of Eq. (20) can be obtained by writing the operand of the logarithmic term as

$$\left[\frac{\left(1+\frac{T_1}{T_2}Ntu\right)\left(1+\frac{T_2}{T_1}Ntu\right)}{(1+Ntu)^2}\right] = 1+\frac{Ntu}{(1+Ntu)^2}\left(\frac{T_2}{T_1}-1\right)^2. \tag{22}$$

For small values of $\frac{Ntu}{(1+Ntu)^2}(\frac{T_2}{T_1}-1)^2$, valid for many applications, a single term in the series expansion is adequate, giving

$$N_{s1}=\frac{T_1\dot{S}_{gen}}{\dot{Q}}\approx(1-\varepsilon)\left(1-\frac{T_1}{T_2}\right). \tag{23}$$

The closeness of this linearisation is evident in Fig. 3(b), especially for low $T_2/T_1$.

The design problem is simplified somewhat by writing the dimensionless heat flow as

$$B=\dot{Q}/\dot{m}c_pT_1=\varepsilon(T_2/T_1-1), \tag{24}$$

since this is a function of specified process parameters, and expressing the temperature ratio $T_2/T_1$ in terms of $B$ and $\varepsilon$. The new relationship then becomes

$$N_{s1}=\frac{1}{B}\ln\left((1+B)\left(1-\varepsilon\frac{B}{(\varepsilon+B)}\right)\right). \tag{25}$$

This is shown in Fig. 5.

As mentioned by Witte and Shamsundar, the exergy loss is simply obtained by multiplying $N_{s1}$ by $T_0/T_1$. Witte and Shamsundar also observed that in many cases (e.g., Brayton cycle recuperators, feed preheat trains) the cold inlet temperature $T_1$ is very close to the environmental temperature $T_0$, thus justifying the corresponding simplification in their analysis. The generality of $T_0 \neq T_1$ is retained in the present work.

A further observation on the present formulation (Eq. (21)) is that $N_{s1}$ can exceed unity. This corresponds to the point made by Bejan [10] that Witte and Shamsundar's thermodynamic efficiency parameter $\eta = 1-T_0\dot{S}_{gen}/\dot{Q}$ can be negative in cryogenic operational conditions — a conceptually inconvenient result.

### 3.2. General analysis for exchangers with flow imbalance

In this case, characterised by $\dot{m}_1 \neq \dot{m}_2$, with the ratio $(c\dot{m})_{max}/(c\dot{m})_{min}=C_{max}/C_{min}$ denoted by $C^*$, the entropy

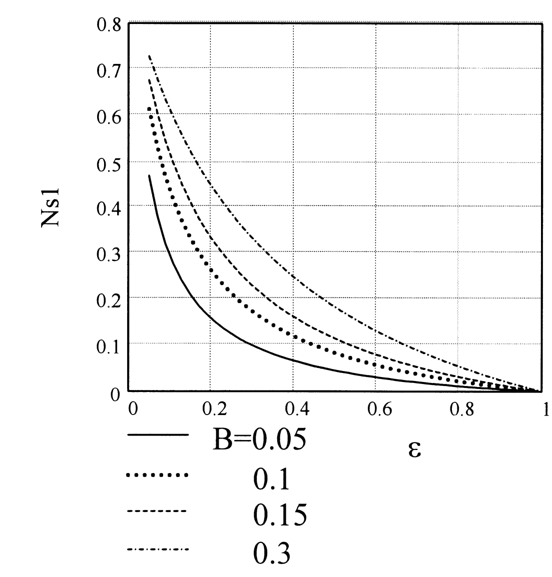


Fig. 5. Entropy generation number $N_{s1}$ in terms of parameter $B$.

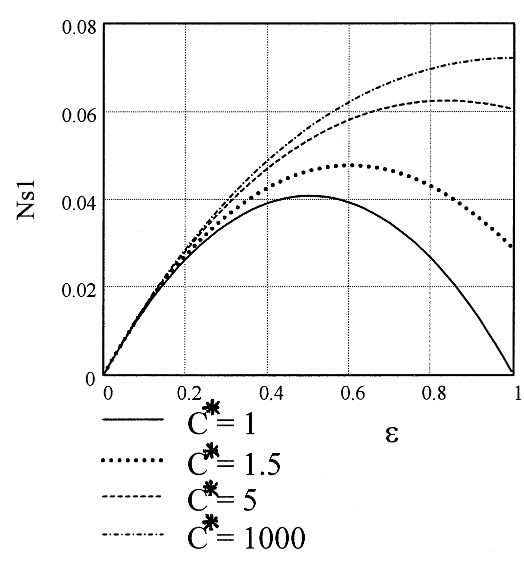


Fig. 6. Bejan's entropy generation number $N_s$, for (1) $C^*=1$, (2) $C^*=1.5$, (3) $C^*=5$, (4) $C^*=1000$.

generation rate becomes [1], again neglecting the pressure drop contribution:

$$\dot{S}_{\mathrm{gen}} = C_{\min}\ln\left(\frac{T_{1\mathrm{out}}}{T_1}\right) + C_{\max}\ln\left(\frac{T_{2\mathrm{out}}}{T_2}\right), \tag{26}$$

for the case with stream 2 having the larger heat capacity rate, giving

$$\frac{\dot{S}_{\mathrm{gen}}}{C_{\min}} = \ln\left[1 + \varepsilon\left(\frac{T_2}{T_1} - 1\right)\right] + C^*\ln\left[1 - \frac{\varepsilon}{C^*}\left(1 - \frac{T_1}{T_2}\right)\right], \tag{27}$$

where

$$\varepsilon = \frac{T_{1\mathrm{out}} - T_1}{T_2 - T_1} = C^*\frac{T_2 - T_{2\mathrm{out}}}{T_2 - T_1}. \tag{28}$$

This is the entropy generation number $N_s$ [11], and is shown in Fig. 6 for $C^* = 1$, 1.5, 5 and 1000.

The entropy generation number $N_{s1}$ then becomes

$$N_{s1} = \frac{T_1\dot{S}_{\mathrm{gen}}}{\dot{Q}} = \frac{T_1}{\varepsilon(T_2 - T_1)}\left\{\ln\left[1 + \varepsilon\left(\frac{T_2}{T_1} - 1\right)\right] + C^*\ln\left[1 - \frac{\varepsilon}{C^*}\left(1 - \frac{T_1}{T_2}\right)\right]\right\}, \tag{29}$$

where

$$\dot{Q} = C_{\min}(T_{1\mathrm{out}} - T_1) = C_{\max}(T_2 - T_{2\mathrm{out}}) = \varepsilon C_{\min}(T_2 - T_1). \tag{30}$$

Note that in the limit of the balanced counterflow case $C = 1$, this reduces to Eq. (13) with $\varepsilon = Ntu/(1 + Ntu)$. For the case of stream 1 having the larger heat capacity, the corresponding equation is

$$N_{s1} = \frac{T_1\dot{S}_{\mathrm{gen}}}{\dot{Q}} = \frac{T_1}{\varepsilon(T_2 - T_1)}\left\{C^*\ln\left[1 + \frac{\varepsilon}{C^*}\left(\frac{T_2}{T_1} - 1\right)\right] + \ln\left[1 - \varepsilon\left(1 - \frac{T_1}{T_2}\right)\right]\right\}, \tag{31}$$

with

$$\dot{Q} = C_{\max}(T_{1\mathrm{out}} - T_1) = C_{\min}(T_2 - T_{2\mathrm{out}}). \tag{32}$$

In Figs. 7 and 8 Eqs. (29) and (31) are shown for $C^* = 1.0$, 1.5, 2 and 1000 and $T_2/T_1 = 1.2$ and 1.5 in terms of effectiveness, illustrating the effect of whether the hot or cold stream have the highest heat capacity rate. All cases exhibit the lower limit of $N_{s1} = 1 - T_1/T_2$, and the curves shown correspond to the complement of Witte and Shamsundar's [4] efficiency parameter with $T_1 = T_0$. It is clear that imbalance — either way — increases entropy generation, and that only in the balanced case does $N_{s1}$ approach zero in the limit $\varepsilon \rightarrow 1$. The thermodynamic advantage of the hot stream having the highest heat capacity rate is also evident, as was observed by Witte and Sham-

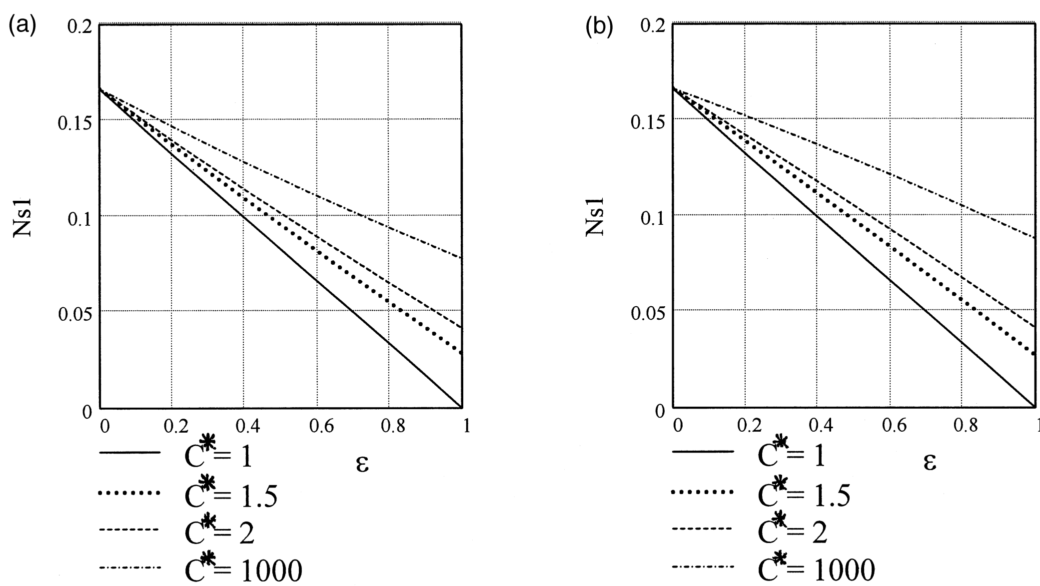


Fig. 7. (a) $N_{s1}$ versus effectiveness, $T_2/T_1 = 1.2$ (hot stream highest $\dot{m}_p$). (b) $N_{s1}$ versus effectiveness, $T_2/T_1 = 1.2$ (cold stream highest $\dot{m}c_p$).

sundar for their efficiency approach. Physically, this reflects the higher mean temperature of heat exchange.

Since the above formulations are in terms of the thermal effectiveness $\varepsilon$, they are perfectly general, and are independent of exchanger flow arrangements. Specific common arrangements are considered briefly below, in terms of the practically useful *Ntu*.

### 3.3. Unbalanced counterflow

The $\varepsilon$–*Ntu* relationship for this case is given by

$$\varepsilon = \frac{1 - \exp\left(-Ntu\left(1 - \frac{1}{C^*}\right)\right)}{1 - \frac{1}{C^*}\exp\left(-Ntu\left(1 - \frac{1}{C^*}\right)\right)}, \tag{33}$$

giving $N_{s1}$–*Ntu* relationships from Eq. (29) as shown in Fig. 9, for temperature ratios $T_2/T_1$ of 1.2 and 1.5, for values of $C^*$ of 1, 1.5, 2 and 1000, the latter approximating to the condensing case of $C^* = \infty$.

### 3.4. Cocurrent (parallel) flow

The $\varepsilon$–*Ntu* relationship for this case is given by

$$\varepsilon = \frac{1 - \exp\left(-Ntu\left(1 + \frac{1}{C^*}\right)\right)}{1 + \frac{1}{C^*}}, \tag{34}$$

and the corresponding $N_{s1}$–*Ntu* relationship is shown in Figs. 10 and 11, for the temperature ratios $T_2/T_1 = 1.2$ and 1.5. It is clear that the effect of flow imbalance is minimal, reflecting the thermodynamic similarity of this configuration to that of condensation, with $C^* = \infty$ (see below).

### 3.5. Condensing on one side

For condensation, with $C^* = \infty$, the $\varepsilon$–*Ntu* relationship is particularly simple:

$$\varepsilon = 1 - \exp(-Ntu). \tag{35}$$

Eq. (29) simplifies to

$$N_{s1} = \frac{\ln\left(1 + \varepsilon\left(\frac{T_2}{T_1} - 1\right)\right)}{\varepsilon\left(\frac{T_2}{T_1} - 1\right)} - \frac{T_1}{T_2}, \tag{36}$$

and allows a relatively simple expression in terms of *Ntu* [12]:

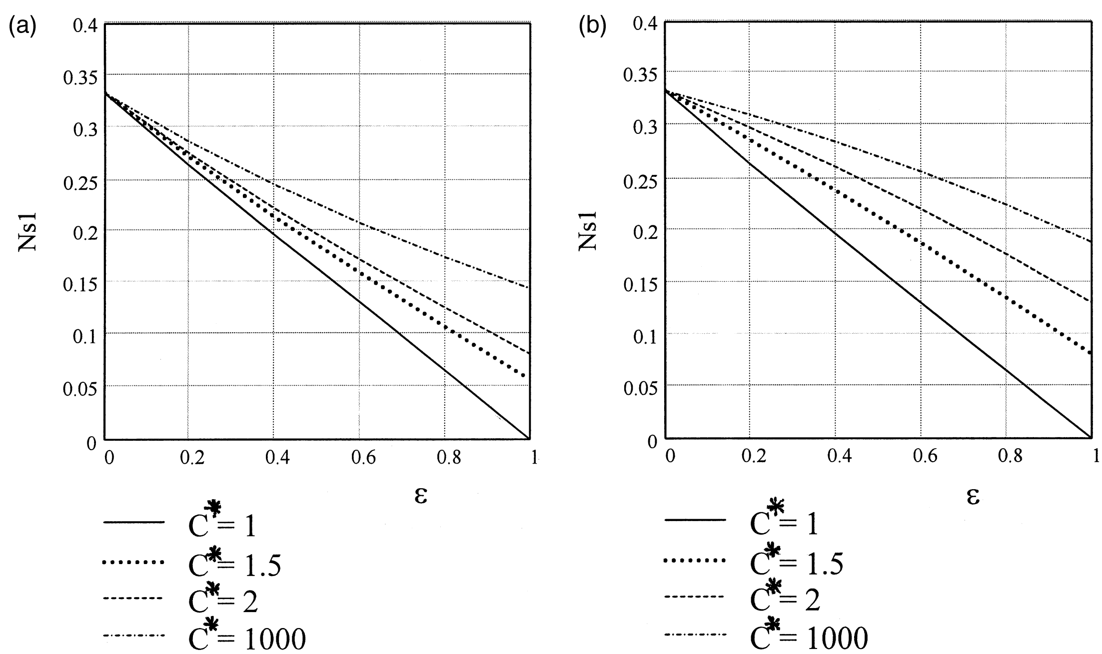


Fig. 8. (a) $N_{s1}$ versus effectiveness, $T_2/T_1 = 1.5$ (hot stream highest $\dot{m}c_p$). (b) $N_{s1}$ versus effectiveness, $T_2/T_1 = 1.5$ (cold stream highest $\dot{m}c_p$).

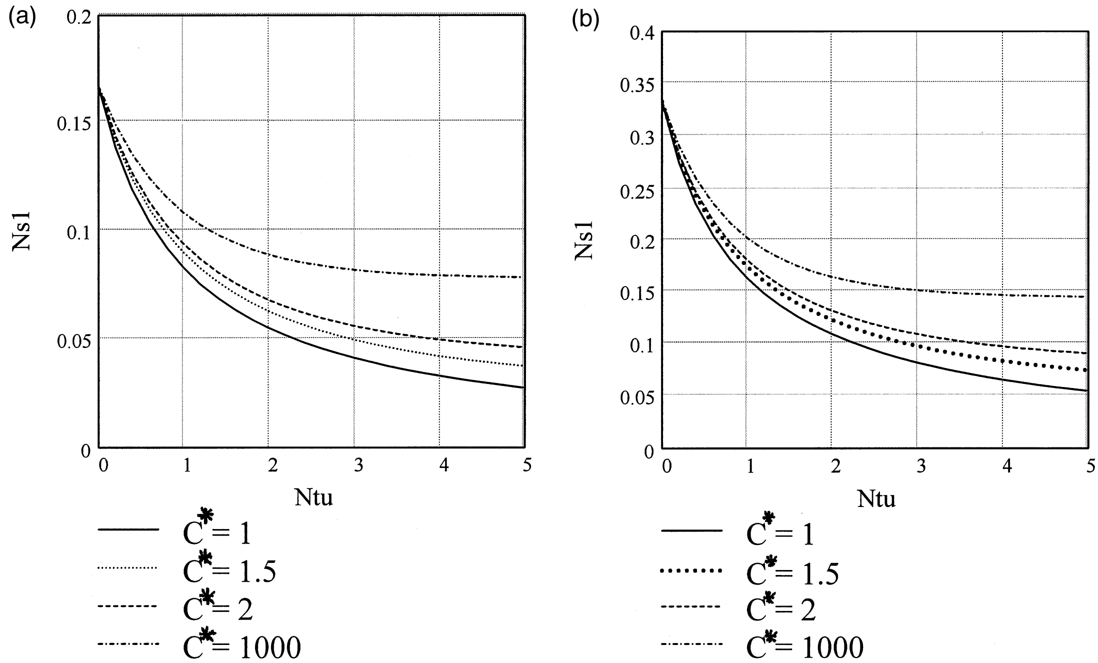


Fig. 9. (a) $N_{s1}$ versus $Ntu$, $T_2/T_1 = 1.2$ (hot stream highest $\dot{m}c_p$). (b) $N_{s1}$ versus $Ntu$, $T_2/T_1 = 1.5$ (hot stream highest $\dot{m}c_p$).

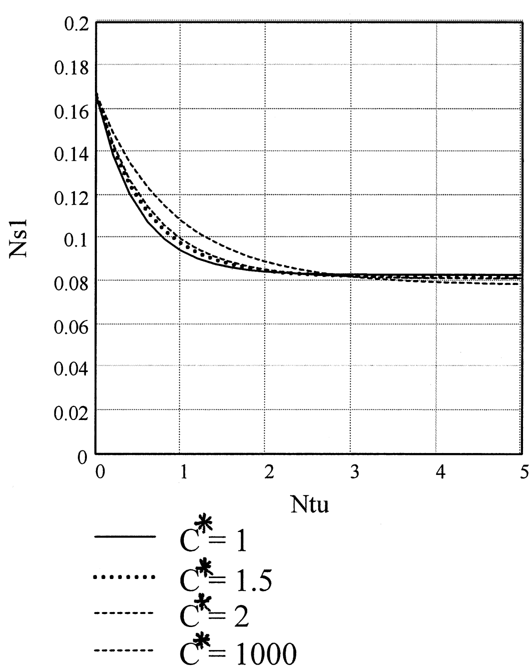


Fig. 10. $N_{s1}$ versus $Ntu$, $T_2/T_1 = 1.2$ (cocurrent flow).

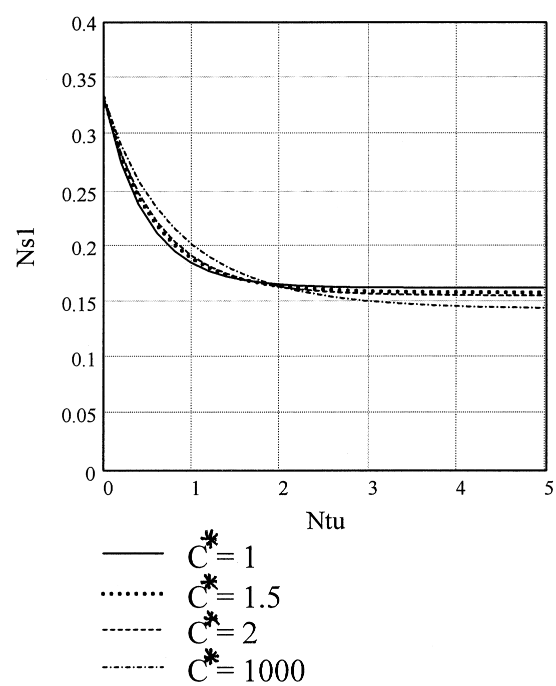


Fig. 11. $N_{s1}$ versus $Ntu$, $T_2/T_1 = 1.5$ (cocurrent flow).

$$N_{s1} = \frac{\ln\left(1 + (1 - \exp(-Ntu))\left(\frac{T_2}{T_1} - 1\right)\right)}{(1 - \exp(-Ntu))\left(\frac{T_2}{T_1} - 1\right)} - \frac{T_1}{T_2}. \quad (37)$$

This is shown in Fig. 12.

### 3.6. Evaporation on one side

For this case $C^* = 0$, the effectiveness relation is the same as for condensation, Eq. (35), and Eq. (31) gives on substitution

$$N_{s1} = \frac{\ln\left[1 - \varepsilon\left(1 - \frac{T_1}{T_2}\right)\right]}{\varepsilon\left(\frac{T_2}{T_1} - 1\right)} + 1, \quad (38)$$

and in terms of $Ntu$:

$$N_{s1} = \frac{\ln\left(1 - (1 - \exp(-Ntu))\left(1 - \frac{T_1}{T_2}\right)\right)}{(1 - \exp(-Ntu))\left(\frac{T_2}{T_1} - 1\right)} + 1. \quad (39)$$

This is shown in Fig. 13. Comparing Figs. 12 and 13, the strong difference between the two cases of condensation and evaporation as the temperature ratio $T_2/T_1$ is increased is clearly seen.

## 4. Finite pressure drop

### 4.1. Optimisation based on local rate equation

We start with the basic Eq. (5) [14,11] for entropy production rate at a given point in the heat exchanger surface with a bulk temperature $T$. A single stream only is examined.

Non-dimensionalising by $q'/T$ gives

$$N_{s1} = \frac{T\dot{S}'_{gen}}{q'} = \frac{\tau}{1+\tau} + \frac{\dot{m}}{\rho q'}\left(-\frac{dp}{dx}\right). \quad (40)$$

Substituting from standard equations for heat transfer surface parameters [14] gives

$$N_{s1} = \frac{fRe^2}{32St^3}\left[\frac{q'\eta}{\dot{m}\rho(c_p T)^{3/2}}\right]^2 \tau^{-3} + \frac{\tau}{1+\tau}. \quad (41)$$

Bejan [14,9] then treats the square bracket term as constant, on the basis that the heat rate $q'$ is specified as constant. The equation is then differentiated with respect to $\tau$ to obtain a minimum in terms of Bejan's

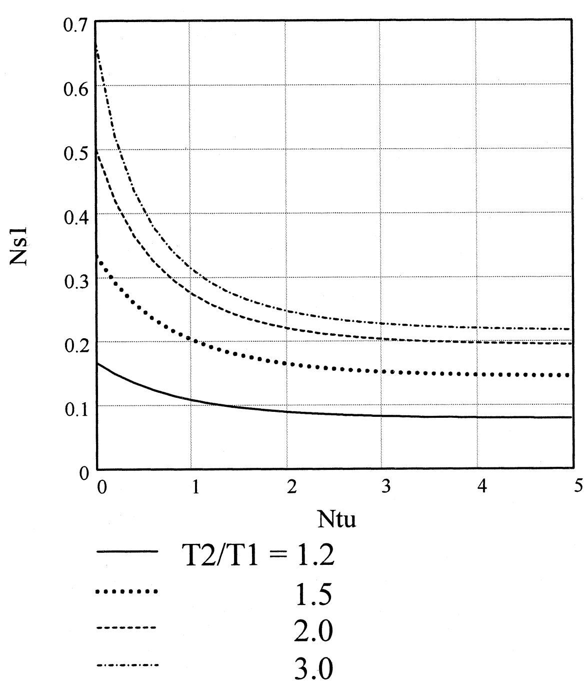


Fig. 12. $N_{s1}$ for condensation on one side ($T_2/T_1 = 1.2, 1.5, 2.0, 3.0$).

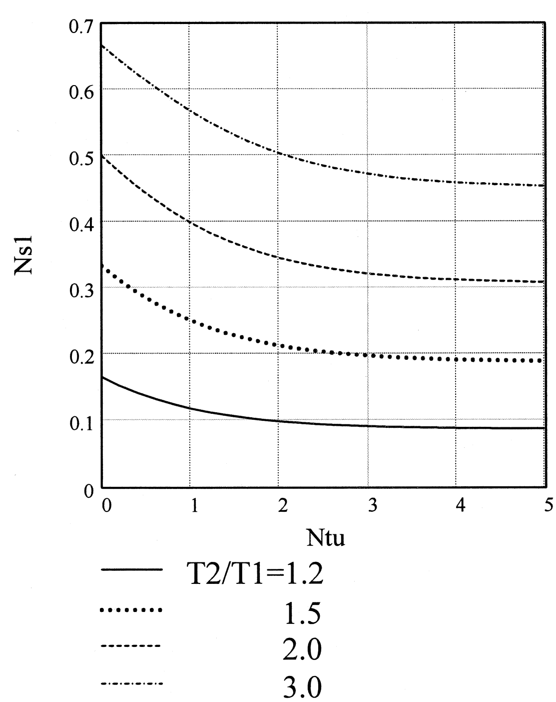


Fig. 13. $N_{s1}$ for evaporation on one side ($T_2/T_1 = 1.2, 1.5, 2.0, 3.0$).

parameter A. Although $q'$ is constant (e.g., with length), however, it is not independent of $\tau$, as is now shown:

$$q' = \alpha p_s \Delta T, \tag{42}$$

where $p_s$ = perimeter of surface, giving

$$q' = St c_p G p_s \Delta T = St G c_p T p_s \tau. \tag{43}$$

Thus a given $q'$ simply relates $G$ to $\Delta T$ and $\tau$, and $p_s$ for the local condition, and the $\tau$ embedded in $q'$ must remain in the equations in order to optimise the contributions from $\tau$ and pressure drop. Mathematically, we could re-write Eq. (41) as

$$N_{s1} = f_1(Re) f_2(Re, \tau)\tau^{-3} + \frac{\tau}{1+\tau}, \tag{41a}$$

Physically this form of relationship arises because the temperature difference is still the driving potential for heat transfer. An important implication of this is that if the heat rate is fixed then $\tau$ can only be optimised if the mass flow, or $G$, is allowed to vary: in fact $G$ becomes a function of $\tau$ on optimisation, as we shall show, with $p_s$, or the hydraulic diameter still available as a variable. The square bracket term in Eq. (41) is

$$[\,] = \frac{\eta St c_p G p_s \Delta T}{\dot{m}\rho \left(c_p T\right)^{3/2}} = J \quad \text{(after Bejan)} \tag{44}$$

$$= \frac{4\eta St \tau}{\rho d_h c_p^{1/2} T^{1/2}}. \tag{45}$$

Thus $J$ is a function of $\tau$, the other variables being functions of local state conditions or Reynolds number. Re-expressing Eq. (41) in terms of $\tau$,

$$N_{s1} = \left(\frac{f Re^2 \eta^2}{2\rho^2 d_h^2 c_p T St}\right)\frac{1}{\tau} + \frac{\tau}{1+\tau}. \tag{46}$$

We can now validly differentiate for minimum $N_{s1}$ because the ( ) term is only a function of $Re$ and $T$. Expressing this term as $A_2$, we have

$$N_{s1} = \frac{A_2}{\tau} + \frac{\tau}{1+\tau}, \tag{47}$$

and differentiating for fixed Reynolds number gives a minimum at a value of $\tau$ denoted by $\tau_{opt}$

$$\tau_{opt} = \frac{A_2^{1/2} + A_2}{1 - A_2} \approx A_2^{1/2} \tag{48}$$

for the normally small $A_2$.

The minimum value of $N_s$ then becomes

$$N_{s1,\,min} = \frac{2A_2^{1/2}}{1 + A_2^{1/2}} + \frac{A_2\left(1 - A_2^{1/2}\right)}{1 + A_1^{1/2}} \approx 2A_2^{1/2} = 2\tau_{opt} \tag{49}$$

for small $A_2$.

These corrected optimum values of $N_{s1,\,min}$ are smaller, by the factor $2/3^{1/2} = 1.1547$, than those obtained by Bejan of

$$\tau_{opt,\,Bejan} = (3A_2)^{1/2} = 3^{1/2}\tau_{opt} = 1.732\tau_{opt}, \tag{50}$$

a substantial correction in the optimum condition, which gives

$$N_{s1,\,min,\,Bejan} = \frac{4}{3^{1/2}}(A_2)^{1/2}, \quad \text{or} \quad N_{s1,\,min\,Bejan} = \frac{4\tau_{opt,\,Bejan}}{3}, \tag{51}$$

using the relationship of the current parameter $A_2$ to Bejan's $A_1$ of

$$A_2 = \frac{A_1^2}{3\tau^2}. \tag{52}$$

Implicit in Eq. (49) is that contributions from the pressure drop and heat flow are equal at the optimum condition of $Be$ = 0.5, which is consistent with Bejan's analysis for developing plate flow ([9,13] and indirectly for counterflow heat exchangers [1]). The plate flow has a direct analogy with that of an offset plate fin heat exchanger surface.

Looking further at the parameter $A_2$,

$$A_2 = \frac{f}{2St}\frac{G^2}{\rho^2 c_p T} = \frac{f}{St}\frac{v^2}{2c_p T}. \tag{53}$$

For a perfect gas, the speed of sound a is given by

$$a^2 = (\gamma - 1)c_p T, \tag{54}$$

so Eq. (53) becomes

$$A_2 = \frac{f}{j} Pr^{2/3}\frac{\gamma - 1}{2}M^2 = \frac{f}{St}\frac{\gamma - 1}{2}M^2, \tag{55}$$

where $M$ is the Mach number.

Noting that $\frac{\gamma-1}{2}M^2$ is the incremental stagnation temperature due to velocity, since $\frac{T_s}{T} = 1 = \frac{\gamma-1}{2}M^2$ for compressible one-dimensional flow, with $T_s$ being the stagnation temperature, the factor $f/St$ in $A_2$ is a simple multiplier in this increment. It is readily shown that the present analysis is consistent with a development of the analysis of Shapiro [15], presented in log-differential form, for a one-dimensional duct flow with friction and heat addition, in which the Reynolds analogy of $f/St = 2$ is assumed.

We now have

$$N_{s1,\,min} = 2M\left(\frac{f}{j}\frac{Pr^{2/3}(\gamma-1)}{2}\right)^{1/2}. \quad (56)$$

Thus $N_{s1,\,min}$ is only related to the area goodness factor $j/f$ and Mach number, for a given gas. For typical gas side velocities of the order of a few m/s, and speed of sound of say 200–300 m/s, a typical Mach number is of the order of 0.01, which gives a corresponding $N_{s1,\,min}$ of the same order. The corresponding temperature difference is then a few degrees, being of order $(0.01{*}T$ K). An alternative form of the minimum value, from Eqs. (49) and (53), is

$$N_{s1,\,min} = 2\left(\frac{fPr^{2/3}}{j}\frac{\dot{m}^2}{2\rho^2 c_p T}\right)^{1/2}\frac{1}{A_c}. \quad (57)$$

The general rate equation, from Eq. (47) is, in terms of Mach number,

$$N_{s1} = \frac{T\dot{S}_{gen}}{q'} = \frac{f}{St}\frac{\gamma-1}{2}M^2\frac{1}{\tau} + \frac{\tau}{1+\tau}. \quad (58)$$

For given process requirements of $\dot{m}$, $\rho$, and heating load rate $q'$, it is clear, as observed by Bejan [1], that the minimum local entropy generation rate *relative to* $q'$ can be made indefinitely small, by making the flow area $A_c$ large enough (or the mass velocity small enough), which simultaneously reduces both the pressure drop and $\Delta T$. The absolute generation rate is proportional to $q'$. More generally, the above analysis shows that the Mach number is the fundamental controlling parameter. Values for typical practical variations of $A_2$ and $\tau$ are shown in Fig. 14, from Eq. (47). Bejan's variations, based on Eq. (41), are shown as a surface plot in Fig. 15 for comparison.

It is clear from Eqs. (56) and (57) that the shape of the $N_{s1,\,min}$ locus is linear with $M$ or $1/A_c$ if $f/j$ is constant, that is, independent of $A_c$ and hence $Re$. In general $f/j$ is a (usually weak) function of $Re$, as already remarked.

### 4.2. Application to single tube heat transfer

The form of Eqs. (56)–(58) is such that they are independent of the type of surface (or duct) involved, and thus are valid directly for application to a single duct or tube. For the circular tube, the only difference now is that the flow area is simply dependent on the tube (= hydraulic) diameter, in contradistinction to the arbitrary duct case in which hydraulic diameter is independent of flow area. This then applies a direct coupling between flow area and temperature difference (for a given heat flow/unit length, as analysed by Bejan [16]), which gives an extra constraint and 'skews' the optimum distribution of entropy generation.

The basic equation for a circular tube is (Bejan):

$$S' = \frac{q'^{\,2}}{\pi Nu\lambda T^2} + \frac{\dot{m}^2 f}{\pi^2\rho^2 T r_o^2} \quad (59)$$

Expressed in the form of the entropy parameter $N_{s1}$,

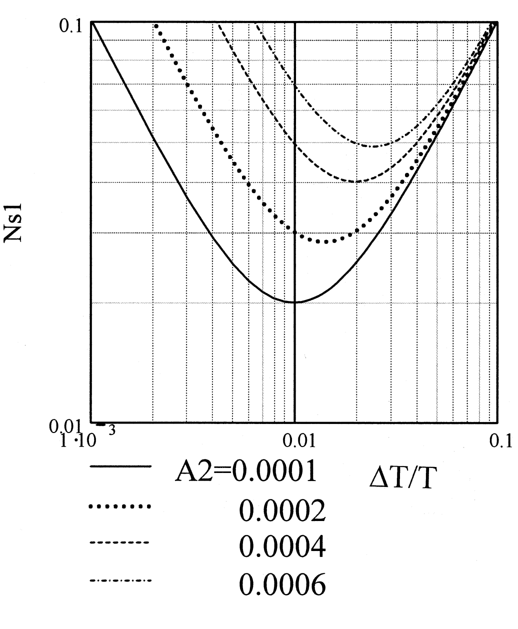


Fig. 14. Parameter $N_s$ versus $\tau$ and $A_2$.

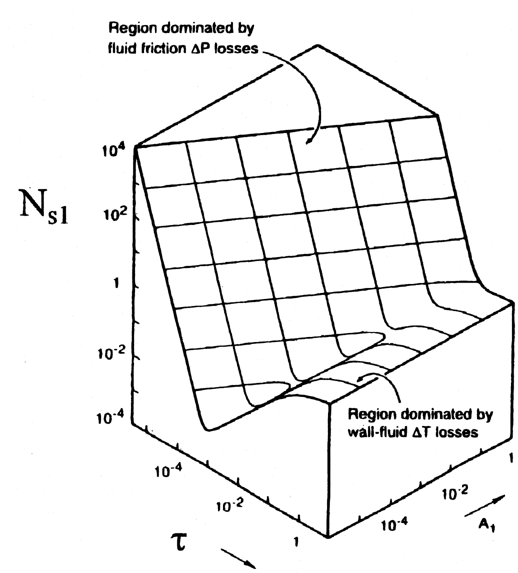


Fig. 15. $N_s$ versus $\tau$ and Bejan's parameter $A$.

this becomes, with some manipulation,

$$N_{s1} = \frac{TS'}{q'} = \frac{q'}{\pi Nu\lambda T} + \frac{B_1 Re^4}{\tau} = \tau + \frac{B_1 Re^4}{\tau}. \quad (60)$$

where

$$B_1 = \frac{F}{N}\frac{\pi^2\eta^5}{32\lambda\rho^2 T\dot{m}^2}, \quad (61)$$

and

$$F = 0.046 \quad (62a)$$

$$N = 0.023Pr^4 \quad (62b)$$

for the commonly-used Dittus–Boelter correlation for turbulent tube flow. But for a fixed axial heat transfer rate $q'$ the Reynolds number and temperature difference are coupled, as indicated above, by

$$q' = \pi Nu\lambda T\tau, \quad (63)$$

giving

$$N_{s1} = \tau + \frac{B_2}{\tau^6}, \quad (64)$$

where

$$B_2 = B_1\left(\frac{q'}{\pi N\lambda T}\right)^5 \quad (65)$$

which is independent of $\tau$ because it implicitly contains the variable $Re$.

Differentiation with respect to $\tau$ and optimisation then gives the optimum $\tau_{opt}$ in terms of $B_2$ and $Re$:

$$\tau_{opt} = (6B_2)^{1/7}. \quad (66)$$

and yields the minimum $N_{s1,\,min}$:

$$N_{s1,\,min} = \tau_{opt}\left(1 + \frac{1}{6}\right) = \left(6^{1/7} + \frac{1}{6^{6/7}}\right)(B_2)^{1/7}. \quad (67)$$

The corresponding optimum Reynolds number is

$$Re_{opt} = \left(\frac{q'}{\pi Nu\lambda\tau T}\right)^{1/0.8}, \quad (68)$$

which gives the same value (10750) as the analysis of Bejan [16] for the conditions given in the example of Bejan of an air flow rate of 100 kg/h at a mean temperature of 1100 K and 1 bar pressure, and with a temperature gradient of 10 K/m. Bejan gives, with a small correction,

$$Re_{opt} = 2.0233Pr^{-0.714}\,Bo^{0.357}, \quad (69)$$

where the parameter $Bo$ is given by

$$Bo = \frac{\rho\dot{m}q'}{\eta^{5/2}(\lambda T)^{1/2}} \quad (70)$$

The distribution ratio of 1 to 1/6 (=0.166) embodied in Eq. (67) is the same as that of Bejan. As seen in Fig. 16, the entropy generation rate is relatively insensitive to $Re$ over quite a wide range of practical interest, for these conditions.

### 4.3. Application of the rate equation to balanced counterflow

The analysis so far is based on a local rate process, and is thus valid for a given point (position, temperature, etc.) in a stream within a heat exchanger. We need now to study in greater depth its application to actual heat exchanger streams with temperature varying along the stream. Considering the rate Eq. (48), still for a single side,

$$\frac{d\dot{S}_{gen}}{dx} = \frac{q'}{T}\left[\frac{A_2}{\tau} + \tau\right] \quad \text{for small } \tau \quad (71)$$

and

$$q' = -\dot{m}c_p\frac{dT}{dx} \quad (72)$$

we have, substituting for $q'$ in Eq. (71),

$$\frac{d\dot{S}_{gen}}{dT} = -\dot{m}c_p\left[\frac{A_2}{\tau} + \tau\right] \quad (73)$$

Recalling that from Eq. (53),

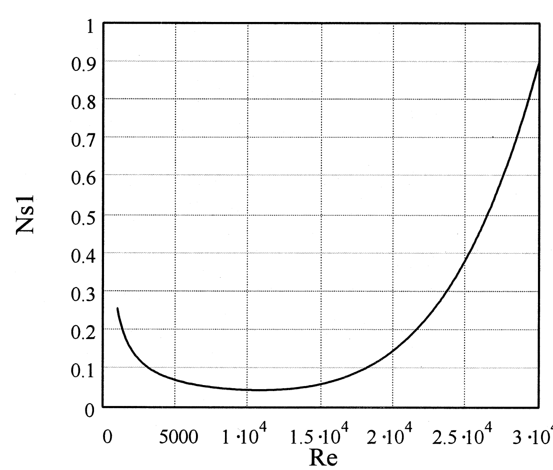


Fig. 16. Variation of single tube $Ns$ with $Re$ for Bejan's [16] example.

$$A_2 = \frac{f}{2St}\frac{G^2}{\rho^2 c_p T} = \frac{f}{2St}\frac{G^2 R^2 T}{p^2 c_p} \quad \text{for a perfect gas.} \tag{74}$$

Here, $G^2$ is fixed by the mass flow rate and through-flow area (which is normally constant) and $f/2St$ is only a weak function (in general of $Re$ (or density). In addition, the variation of absolute pressure $p$ is small in comparison with that of temperature, as shown in Appendix A, so we can put

$$A_2 = AT = \frac{f}{2St}\frac{G^2}{p^2 c_p} R^2 T \tag{75}$$

(where $A$ is very nearly constant),

noting that $A$ has dimensions of $1/T$. Thus we can reformulate (73) as

$$\frac{\mathrm{d}\dot{S}_{\mathrm{gen}}}{\mathrm{d}T} = -\frac{\dot{m}c_p}{T}\left[\frac{AT}{\Delta T} + \frac{\Delta T}{T^2}\right], \tag{76}$$

which now accounts correctly for the density variation.

Thus

$$\dot{S}_{\mathrm{gen}} = -\dot{m}c_p\left[\frac{A}{\Delta T}\int_{T_2}^{T_{2\mathrm{out}}} T\,\mathrm{d}T + \Delta T\int_{T_2}^{T_{2\mathrm{out}}}\frac{dT}{T^2}\right], \tag{77}$$

for side 1, since $\Delta T$ is constant with flow length in a balanced exchanger.

Integrating,

$$\dot{S}_{\mathrm{gen}} = -\dot{m}c_p\left[\frac{A}{2\Delta T}\left(T_1^2 - T_{1\mathrm{out}}^2\right) - \Delta T\left[\frac{1}{T_{1\mathrm{out}}} - \frac{1}{T_1}\right]\right] \tag{78}$$

Putting

$$B_3 = \frac{\dot{m}c_p A}{2}\left(T_1^2 - T_{1\mathrm{out}}^2\right) \tag{79}$$

and

$$C_1 = -\dot{m}c_p\left[\frac{1}{T_{1\mathrm{out}}} - \frac{1}{T_1^2}\right] \tag{80}$$

we have

$$\dot{S}_{\mathrm{gen}} = \frac{B_3}{\Delta T} + C_1\Delta T \tag{81}$$

Differentiation gives a minimum when

$$\Delta T^2 = \Delta T^{*2} = \frac{B_3}{C_1} = \frac{A}{2}(T_1 + T_{1\mathrm{out}})T_1 T_{1\mathrm{out}} \tag{82}$$

and the minimum value is accordingly

$$\dot{S}_{\mathrm{gen,\,min}} = \frac{B_3}{\Delta T^*} + C_1\Delta T^* = 2\sqrt{B_3 C_1} \tag{83}$$

with equal contributions from temperature difference and pressure drop as for the local case.

Note that in the limit of $T_{1\mathrm{out}} \to T_1$, the optimum temperature difference converges to the 'local' value of

$$\Delta T = \tau T = T\sqrt{AT} = TA_2^{1/2}. \tag{84}$$

The expansion of (83) gives

$$\dot{S}_{\mathrm{gen,\,min}} = 2\dot{m}c_p\left(\frac{f}{2St}\frac{G^2 R^2}{p^2 c_p}\right)_2^{1/2}(T_{1\mathrm{out}} - T_1) \times\sqrt{\frac{(T_1 + T_{1\mathrm{out}})}{2(T_1\,T_{1\mathrm{out}})}} \quad \text{(for one side)} \tag{85}$$

and for both sides:

$$\dot{S}_{\mathrm{gen,\,min}} = 2\dot{m}c_p\left(\frac{f}{2St}\frac{G^2 R}{p^2 c_p}\right)_1^{1/2}(T_{1\mathrm{out}} - T_1)\sqrt{\frac{(T_1 + T_{1\mathrm{out}})}{2(T_1 T_{1\mathrm{out}})}} + 2\dot{m}c_p\left(\frac{f}{2St}\frac{G^2 R}{p^2 c_p}\right)_2^{1/2}(T_2 - T_{2\mathrm{out}})\sqrt{\frac{(T_2 + T_{2\mathrm{out}})}{2(T_2\,T_{2\mathrm{out}})}} \tag{86}$$

In the limit of vanishingly small $\Delta T$ (Bejan's [1] condition), $T_{2\mathrm{out}} = T_1$ and $T_{1\mathrm{out}} = T_2$, giving

$$\dot{S}_{\mathrm{gen,\,min}} = 2\dot{m}c_p(T_2 - T_1) \times\sqrt{\frac{(T_1 + T_2)}{2(T_1 T_2)}\left[\left(\frac{f}{2St}\frac{G^2 R}{\rho^2 c_p}\right)_1^{1/2} + \left(\frac{f}{2St}\frac{G^2 R}{\rho^2 c_p}\right)_2^{1/2}\right]} \tag{87}$$

Bejan's corresponding relationship for both sides is

$$\dot{S}_{\mathrm{gen,\,min}} = \frac{2\dot{m}c_p(T_2 - T_1)}{\sqrt{T_1\,T_2}}\left[\left(\frac{f}{2St}\frac{G^2 R}{p^2 c_p}\right)_1^{1/2} T_{1\mathrm{ref}}^{1/2} + \left(\frac{f}{2St}\frac{G^2 R}{p^2 c_p}\right)_2^{1/2} T_{2\mathrm{ref}}^{1/2}\right] \tag{88}$$

where $T_{1\mathrm{ref}}$ and $T_{2\mathrm{ref}}$ are the implied reference temperatures pertaining to the definition by Bejan of dimensionless mass velocity $g_1$ as $g_1 = \frac{G}{(2\rho p)^{1/2}}$. It is clear from Appendix A that the density is closely proportional to the inverse of absolute temperature, whilst the relative variation of pressure is small, so that $g_1$ is not constant along the exchanger as assumed by Bejan. Inspection of Eqs. (87) and (88) shows that Bejan's relationship and the more general one are in agreement if $T_{1\mathrm{ref}} = T_{2\mathrm{ref}} = (T_1 + T_2)/2$, by Bejan's assumption of infinitesimal $\Delta T$, or in other words, if the density in $g_1$ is selected at the arithmetic mean temperature of both streams — a felicitous result.

For one side, the consequences of evaluating the

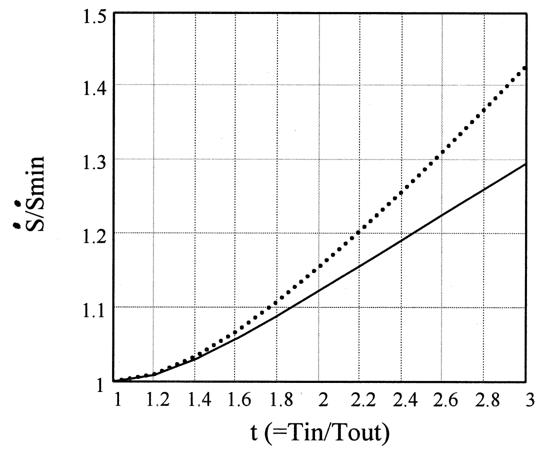


Fig. 17. Relative entropy generation for evaluating Eq. (84) at terminal temperatures.

entropy rate at a single terminal temperature instead of using the full optimum relationship are shown in Fig. 17, in terms of the variable $t = T_{in}/T_{out}$ of the hot stream.

The present analysis thus reflects the relaxation of the requirement for constant $g_1$, and the optimum formulations given by Eqs. (85) and (86) are applicable for both non-vanishing $\Delta T$ and variable density. They are thus more accurate for optimising in the case of real balanced counterflow exchangers with 'long' duties, i.e. high temperature span.

## 5. Concluding remarks

An attempt has been made in this work to resolve the 'entropy generation paradox' identified by Bejan, by consideration of the basis of non-dimensionalising the entropy generation rate. It is shown that the approach introduced by Witte and Shamsundar of using the heat flow to non-dimensionalise enables a unification of the approaches and a clarification of the paradox. The characteristic temperature used is the lower terminal temperature, and the analysis, initially developed for balanced counterflow, is extended to other flow configurations.

A development of the rate process for a general heat transfer surface has corrected the analysis of Bejan, the new results showing equi-partition of the optimum entropy generation between temperature difference and pressure drop components, being now consistent with other optimised results. This analysis is extended to show that the fundamental controlling parameter for ideal gas flows is the flow Mach number, thus making a formal correspondence with Shapiro's [15] log-differential presentation of entropy generation in one-dimensional duct flow. Application of the process to a single-tube geometry, optimising the temperature difference, yields the same result as Bejan, who optimised for Reynolds number.

Finally, the local rate equation is used to develop a new relationship for optimising balanced counterflow exchangers for a 'long' duty, that is with significant temperature changes of the working fluids. This is shown to be compatible with Bejan's [1] analysis if the reference temperature used in the latter is the arithmetic mean of the terminal temperatures.

Without going to the specialised further stage of incorporating the exergy expended in manufacture of a heat exchanger, on which there have been several studies, clear needs for further work exist in examining the optimisation of unbalanced exchangers with pressure drop included, and also of nearly balanced exchangers with the 'ideal' of constant $\Delta T/T$. The latter will be the subject of a future paper.

## Appendix A. Variation of pressure

The dimensionless increment of absolute pressure $p$ in a given increment of length d$x$ can be written

$$\frac{\Delta p}{p} = -\frac{2f}{d_h}\frac{G^2}{\rho}\frac{dx}{p} \tag{A1}$$

In the same distance, the fluid (assumed to be an ideal gas) has a temperature change $\Delta_1 T$ (to distinguish between it and the driving temperature difference $\Delta T$) given by

$$\frac{\Delta_1 T}{T} = \frac{q'x}{\dot{m}c_p T} = \frac{2St\Delta T\,dx}{Td_h} \tag{A2}$$

But we have shown that at the optimum point, also covering near-optimum conditions,

$$\frac{\Delta T}{T} = \tau = A_2^{1/2} = \left(\frac{f}{2St}\frac{q'^2}{\rho^2 c_p T}\right)^{1/2} \tag{A3}$$

giving, on substitution, and dividing Eq. (A1) by (A2)

$$\left(\frac{\frac{\Delta p}{p}}{\frac{\Delta_1 T}{T}}\right) = -\frac{2\gamma\tau}{\gamma - 1} \tag{A4}$$

or, in terms of Mach number

$$\left(\frac{\frac{\Delta p}{p}}{\frac{\Delta_1 T}{T}}\right) = -\left(\frac{2f}{St}\right)^{1/2}\frac{\gamma}{(\gamma - 1)^{1/2}}M \tag{A5}$$

Thus the relative pressure drop compared with the relative temperature change is of the same order as the dimensionless driving temperature difference $\tau$, or Mach number $M$, near the optimum condition. We can thus say that with this proviso, to a high degree of accuracy, the relative density change is proportional to the relative temperature change.

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